AXIALLY SYMMETRIC LONG WAVES ON THE SURFACE OF A VARYING-DEPTH BASIN

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The axially symmetric Korteweg–de Vries (KdV) equation for the case of a constant-depth basin was obtained in [1]. In the present paper we derive the axially symmetric KdV equation for a varying-depth basin. Conditions are shown for the equation obtained, under which the asymptotic behavior of its solution is described by an equation of the form

$$u_t + uu_x + u_{xxx} = 0,$$

whose asymptotic behavior is well known [2].

1. For an inviscid incompressible heavy liquid the problem of potential motion in a varying-depth basin is exactly formulated (in the planar and axially symmetric cases) as follows:

equation of continuity

$$u_r + w_z + \frac{ku}{r} = 0; \tag{1.1}$$

equation of no vorticity

$$u_z - w_r = 0; \tag{1.2}$$

constancy of pressure at the free boundary

$$z = h(r, t),\ \ u_t + h_r + uu_r + ww_r + h_r w_t - h_t w_r = 0; \tag{1.3}$$

kinematic condition at the free boundary

$$h_t - w + uh_r = 0; \tag{1.4}$$

and the nonflow condition at the bottom

$$z = -H(r),\ w = -\dot{H}u. \tag{1.5}$$

The system of equations (1.1)-(1.5) was written in dimensionless form. All lengths are measured here in units of H_0 (the characteristic basin depth), velocities in units of $(gH_0)^{1/2}$, time in units of $(H_0/g)^{1/2}$, where g is the acceleration due to gravity force, and the vertical z coordinate is measured from the unperturbed free surface of the liquid, corresponding in the axially symmetric case to k = 1, and in the planar case to k = 0. We note that Eq. (1.3) was obtained by the usual Cauchy–Lagrange integral by differentiation with respect to r along the surfaces z = h(r, t) and $\dot{H} \equiv dH/dr$.

Supplementing the system (1.1)-(1.5) by initial conditions, we obtain the full formulation of the problem.

As is well known, the system of equations (1.1)-(1.5) possesses two important conservation laws – mass and energy – which for bounded h, u, and w tending quite quickly to zero for $r \to \infty$, have the following form:

$$\int_0^\infty hr^k dr = C_1; \tag{1.6}$$

$$\int_0^\infty \int_{-H(r)}^{h(r,t)} (u^2 + w^2) r^k dr + \int_0^\infty h^2 r^k dr = C_2, \tag{1.7}$$

where k = 1 corresponds to the axially symmetrical case. In the planar case it is necessary to put k = 0 and note that the lower integration limit over r is $-\infty$.

To derive approximate equations describing wave propagation in one (positive) direction of the r axis it is necessary to determine orders of magnitude. We assume that h, u, w, $\partial/\partial t$, $\partial/\partial r$, $\partial/\partial z$ are of the same order of smallness as in an isolated wave propagating in the positive direction of the r axis. It can be verified that for an isolated wave propagating over a uniform bottom, in a coordinate system moving with critical velocity $V = (gH_0)^{1/2}$ the following estimates hold: If $h \sim \epsilon \ll 1$, then

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$$u \sim \varepsilon,\ w \sim \varepsilon^{3/2}, \partial/\partial t \sim \varepsilon^{3/2},\ \partial/\partial r \sim \varepsilon^{1/2},\ \partial/\partial z \sim 1. \tag{1.8}$$

For the case of a basin with a nonuniform bottom it is natural to assume (at least for the problem of evolution of waves arriving from the region of a uniform bottom) that under some restrictions on the smoothness of the function H(r) and smallness of the bottom slope $\dot{H}(r)$ the estimates (1.8) remain valid in a coordinate system moving with velocity $V = (gH(r))^{1/2}$. The smallness of the bottom slope, required for isolating waves of a single direction, can be estimated as follows. In the linear approximation the propagation of axially symmetric waves is described by the system of equations

$$h_t + Hu_r + \dot{H}u + (1/r)Hu = 0; \tag{1.9}$$
$$u_t + h_r = 0. \tag{1.10}$$

A wave moving in the positive direction must have the form

$$h = \varphi_1(r)\, h(\xi) + O(\varepsilon),\quad u = \varphi_2(r)\, u(\xi) + O(\varepsilon),\quad \xi = \int\limits_{r_0}^{r} H^{-1/2} dr - t. \tag{1.11}$$

It can be shown that in the general case of an arbitrary function H(r) the solution of system (1.9), (1.10) has the form (1.11) when and only when the last two terms of the left-hand side of Eq. (1.9) are successively of small order relative to the first two terms, i.e, when they can be neglected in the zeroth approximation. In this case the reflected waves will be small in amplitude, and in the zeroth approximation will not affect wave propagation in the positive direction.

Consequently, separation of waves of one direction is possible for $\dot{H}(r) \leqslant \varepsilon^{3/2}$, $r \geqslant \varepsilon^{-3/2}$ (the length of waves under consideration being $\varepsilon^{-1/2}$). Thus, the orders of all quantities were determined, and we can now turn to deriving the approximate equation.

2. We introduce new unknown functions and independent variables by the following relations:

$$h = \varepsilon h',\ u = \varepsilon u',\ w = \varepsilon^{3/2} w',\ r = \varepsilon^{-1/2} r',\ t = \varepsilon^{-1/2} t'; \tag{2.1}$$

$$x = \int\limits_{r_0}^{r'} H^{-1/2} dr' - t',\quad \tau = \varepsilon r',\quad z = [\varepsilon h' - H(\tau)]\, q - H(\tau), \tag{2.2}$$

where x, q, and τ are new independent variables.

The replacement of independent variables (2.2) implies transition to a coordinate system moving in the positive direction of the r axis with critical velocity $V = (H(\tau))^{1/2}$. Moreover, we transform from a region of motion $r < \infty$, $-H(\tau) \leqslant z \leqslant h(x,\ t)$ with unknown upper boundary to a fixed q region $x < \infty$, $0 \leqslant q \leqslant 1$.

Substituting (2.1), (2.2) into (1.1)-(1.5) and omitting the primes, we obtain

$$\varepsilon^{3/2}(u_x + H^{-1/2} w_q) + \varepsilon^{5/2}[H^{1/2} u_\tau + H^{-1} h u_x - (H^{-1} h_x + H^{-1/2}\dot{H}) q u_q + k H^{1/2} u/\tau] = O(\varepsilon^{7/2}); \tag{2.3}$$
$$\varepsilon u_q + \varepsilon^2 H^{1/2} w_x = O(\varepsilon^3); \tag{2.4}$$
$$\varepsilon^{3/2}(h_x - H^{1/2} u_x) + \varepsilon^{5/2}(H^{1/2} h_\tau + H^{-1} h h_x - H^{-1/2} h u_x + u u_x) = O(\varepsilon^{7/2}); \tag{2.5}$$
$$\varepsilon^{3/2}(h_x + w) - \varepsilon^{5/2} H^{-1/2} u h_x = O(\varepsilon^{7/2}) \quad \text{for } q = 1; \tag{2.6}$$
$$\varepsilon^{3/2} w + \varepsilon^{5/2} \dot{H} u = 0,\ H = H(\tau),\ \dot{H} \equiv dH/d\tau \quad \text{for } q = 0. \tag{2.7}$$

We now seek a solution of system (2.3)-(2.7) in the form of a power series in

$$h = \sum_{n=0}^{\infty} h_n(x, \tau)\, \varepsilon^n,\quad u = \sum_{n=0}^{\infty} u_n(x, \tau, q)\, \varepsilon^n,\quad w = \sum_{n=0}^{\infty} w_n(x, \tau, q)\, \varepsilon^n. \tag{2.8}$$

Substituting (2.8) into (2.3)-(2.7) and equating to zero the coefficient of the lowest power of ϵ in each of Eqs. (2.3)-(2.7), we obtain the zeroth approximation problem

$$u_{0x} + H^{-1/2} w_{0q} = 0,\ u_{0q} = 0;$$

for $q = 1$ $h_{0x} - H^{1/2} u_{0x} = 0,\ h_{0x} + w_0 = 0;$
for $q = 0$ $w_0 = 0$; for $x \to \infty$ $h_0 = u_0 = w_0 = 0.$

The solution of this problem is

$$u_0 = H^{-1/2} h_0,\ w_0 = -h_{0x} q; \tag{2.9}$$

with $h_0(x, \tau)$ remaining undetermined. To obtain an equation for $h_0(x, \tau)$ it is necessary to consider the following approximation.

Taking into account (2.9), this approximation gives the problem

$$u_{1x} + H^{-1/2}w_{1q} + h_{0\tau} + H^{-3/2}h_0h_{0x} - \left(\frac{1}{2}H^{-1}\dot{H} - \frac{k}{\tau}\right)h_0 = 0; \tag{2.10}$$

$$u_{1q} + H^{1/2}qh_{0xx} = 0; \tag{2.11}$$

$$h_{1x} - H^{1/2}u_{1x} + H^{1/2}h_{0\tau} + H^{-1}h_0h_{0x} = 0,\ h_{1x} + w_1 - H^{-1}h_0h_{0x} = 0 \quad \text{for } q = 1; \tag{2.12}$$

$$w_1 + H^{-1/2}\dot{H}h_0 = 0 \text{ for } q = 0. \tag{2.13}$$

The unknown functions u_1 and w_1 are found from Eqs. (2.10), (2.11) with account of (2.13). Substituting u_1 and w_1 just found in the boundary conditions (2.12), we obtain

$$h_{1x} - H^{1/2}\Phi_x + H^{1/2}h_{0\tau} + H^{-1}h_0h_{0x} + \frac{1}{2}Hh_{0xxx} = 0,$$

$$h_{1x} - H^{1/2}\Phi_x - H^{1/2}h_{0\tau} - 2H^{-1}h_0h_{0x} + \frac{1}{6}Hh_{0xxx} - \left(\frac{1}{2}H^{-1/2}\dot{H} + \frac{kH^{1/2}}{\tau}\right)h_0 = 0,$$

where $\Phi(x, \tau)$, generated by integrating Eq. (2.11). Eliminating h_1 from these equations, we obtain the required equation for determining $h_0(x, \tau)$

$$h_{0\tau} + \frac{3}{2}H^{-3/2}h_0h_{0x} + \frac{1}{6}H^{1/2}h_{0xxx} + \left(\frac{1}{4}H^{-1}\dot{H} + \frac{k}{2\tau}\right)h_0 = 0. \tag{2.14}$$

Equations (2.14), (2.9) determine in the zeroth approximation the wave propagating in the positive direction of the r axis in the axially symmetric (k = 1) and planar (k = 0) cases in a varying-depth basin. For $H(\tau) \equiv 1$ the equation coincides with that obtained in [1], where several properties of this equation were investigated. For k = 0 (the planar case) Eq. (2.14) coincides within the accuracy of the coordinate system with the "generalized" KdV equation [3].

Substituting in the conservation laws (1.6), (1.7) the new variables (2.1), (2.2) and taking into account (2.8), (2.9) we obtain

$$\tau^kH^{1/2}(\tau)\int_{-\infty}^{\infty} h_0(\tau, x)\,dx = \tau_0^kH^{1/2}(\tau_0)\int_{-\infty}^{\infty} h_0(\tau_0, x)\,dx = C_1 + O(\varepsilon); \tag{2.15}$$

$$\tau^kH^{1/2}(\tau)\int_{-\infty}^{\infty} h_0^2(\tau, x)\,dx = \tau_0^kH^{1/2}(\tau_0)\int_{-\infty}^{\infty} h_0^2(\tau_0, x)\,dx = C_2 + O(\varepsilon), \tag{2.16}$$

where C_1 and C_2 are constants, and $h_0(\tau_0, x)$ is the initial perturbed free surface. The appearance of $-\infty$ at the lower limit is explained by the fact that the waves propagate with a length of the order of $\epsilon^{-1/2}$ at $r \gtrsim \varepsilon^{-3/2}$ Since the mass and energy conservation laws are among the most important properties of the original exact problem, it is natural to assume that among all solutions of the approximate equation (2.14) only those satisfying the conservation laws (2.15), (2.16) have physical meaning.

It can be verified that the energy conservation law (2.16) is satisfied for any exact solution of Eq. (2.14) under the assumption made above of sufficiently fast limit tendency $h_0(\tau_0, x) \to 0$ for $|x| \to \infty$. The mass conservation law (2.15) is satisfied only when $C_1 = 0$, i.e.,

$$\int_{-\infty}^{\infty} h_0(\tau, x)\,dx = \int_{-\infty}^{\infty} h_0(\tau_0, x)\,dx = 0, \tag{2.17}$$

while the vanishing of the integral of the initial perturbation $h_0(\tau_0, x)$ over x is a necessary and sufficient condition for satisfying the mass conservation law for k = 1 and arbitrary function $H(\tau)$. This firstly implies that in the axially symmetric case there are depressions along with humps in a wave of a single direction. Secondly, it follows that Eq. (2.14) is applicable to the axially symmetric case not earlier than the moment of time when a wave satisfying condition (2.17) is evolved from an arbitrary initial perturbation. Until this moment waves of both directions are important for the evolution of the initial perturbation, and Eq. (2.14) is not sufficient for describing this evolution.

In the planar case (k = 0) the mass conservation law is satisfied at $H(\tau) \equiv$ const for arbitrary C_1, and in the absence of condition (2.17), while for k = 1 this condition is necessary also for $H(\tau) \equiv$ const [1].

It must be noted that in the planar case and $H(\tau) \neq$ const condition (2.17) is not satisfactory, since the existence was proved of a solitary wave, for which (2.17) is not satisfied. In this case, obviously, it is necessary to consider an equation for h_1, allowing the appearance of waves with a length of an order of magnitude larger than for h_0. The presence of these waves, not violating the quadratic energy conservation law (due to smallness of amplitude), can lead to satisfaction of the mass conservation law (2.15). From the physical point of view the appearance of waves with a length of the order of the bottom inhomogeneity is natural. This problem, however, is a separate problem not considered in the present paper.

Taking into account the remarks made, the propagation problem of long planar (k = 0) and axially symmetric (k = 1) waves in the positive direction of the r axis can now be stated as follows: Find a function $h_0(\tau, x)$ satisfying Eq. (2.14), the initial condition $h_0(\tau_0, x) = F(x)$ [F(x) is a given function], the boundary condition $h_0(\tau, x) \to 0$ for $|x| \to \infty$, and the conservation laws (2.16), (2.17). For k = 0 and $H(\tau) \equiv$ const (2.17) is replaced by (2.15).

3. One of the interesting problems in studying the problem stated is that of asymptotic behavior of the solution of Eq. (2.14) for large τ at given $H(\tau)$. In particular, the practical important case of decreasing depth with increasing τ, corresponding to wave migration from an open ocean to a coast, is of interest.

It is seen from Eq. (2.14) that with decreasing depth the role of the nonlinear term must increase due to the coefficient of $H^{-3/2}$. On the other hand, it is well known that the presence of nonlinearity leads to distortion of the front wave, which automatically involves an increase of the dispersive term with a third derivative of h_0 with respect to x. Consequently, this term must be of the same order as the nonlinear one, despite the fact that its coefficient is decreasing. Taking these considerations into account, one can attempt to represent h_0 in the form

$$h_0(x, \tau) = f(\tau)U[\varphi(\tau)x, T(\tau)], \tag{3.1}$$

where the functions $f(\tau)$ and $\varphi(\tau)$ are determined by two conditions. One of them is the equality of coefficients of the nonlinear and dispersive terms. As a second condition we require that after substituting (3.1) in the energy conservation law (2.16) the following relation is obtained:

$$\int_{-\infty}^{\infty} U^2(T, \eta)\, d\eta = \int_{-\infty}^{\infty} U^2(0, \eta)\, d\eta = C, \tag{3.2}$$

where $\eta = \varphi(\tau)x$ and $T(\tau)$ are new independent variables. This leads to representing h_0 in the form

$$h_0(x, \tau) = \frac{1}{\tau^{2k/3}H} U(\eta, T); \tag{3.3}$$

$$\eta = \frac{3x}{\tau^{k/3}H^{3/2}}, \quad T = \frac{9}{2}\int_{\tau_0}^{\tau} \frac{d\tau}{\tau^k H^4}. \tag{3.4}$$

Substituting (3.3), (3.4) into (2.14), (2.15), we obtain

$$U_T + UU_\eta + U_{\eta\eta\eta} - \tau^k H^4 \left(\frac{2}{9}\frac{k}{\tau} + \frac{H'}{H}\right)\left(\frac{1}{3}\eta U_\eta + \frac{1}{6}U\right) = 0; \tag{3.5}$$

$$\int_{-\infty}^{\infty} U(T, \eta)\, d\eta = \int_{-\infty}^{\infty} U(0, \eta)\, d\eta = 0. \tag{3.6}$$

Moreover, $U(T, \eta)$ satisfies the energy conservation law (3.2).

It follows from Eq. (3.5) that in the special case $k = 1$, $H(\tau) = \left(\frac{\tau}{\tau_0}\right)^{-2/9}$ $U(T, \eta)$ is determined by the equation

$$U_T + UU_\eta + U_{\eta\eta\eta} = 0, \tag{3.7}$$

i.e., the KdV equation, whose asymptotic solution for large T was investigated in significant detail [2].

It follows in this case from (3.4) that

$$T = \frac{81}{16}\left[\left(\frac{\tau}{\tau_0}\right)^{8/9} - 1\right] = \frac{81}{16}\left[H^{-4} - 1\right],$$

whence it is seen that $T \to \infty$ for $\tau \to \infty$, and, consequently, the asymptotic $U(T, \eta)$ for $T \to \infty$ determines through (3.3), (3.4) the asymptotic $h_0(\tau, x)$ for $\tau \to \infty$. In particular, for initial perturbations satisfying condition (3.6) two characteristic variations of the behavior of the solution of Eq. (3.7) are possible, depending on the parameter σ^2, equal to the ratio of the nonlinear term to the dispersive term at the initial moment. For σ values less than some critical σ_0 the solution is represented in the form of a spreading wave packet, similar in velocity of decreasing amplitude and phase motion to the solution of the linearized KdV equation. For $\sigma > \sigma_0$, the solution at $\tau \to \infty$ is represented as one of several solutions running to the right:

$$U_i(T, \eta) = a_i \operatorname{ch}^{-2}\left[\sqrt{\frac{a_i}{12}}\left(\eta - \frac{a_i}{3} T\right)\right], \tag{3.8}$$

and the wave packet spreads in the region $\eta \leqslant 0$, with a_i = const, i.e., after their formation and separation of the spreading tail the solitons remain stationary (in the coordinates η, T). It follows from (3.3), (3.4) that in this case the amplitude of solitons (3.8) decreases as $\tau^{-4/9}$, their length is independent of τ, and the migration velocity decreases as $\tau^{-1/9}$.

Let now

$$H(\tau) = (\tau/\tau_0)^{-\alpha}, \ \alpha > 0. \tag{3.9}$$

Substituting (3.9) into (3.4), we obtain

$$T = \frac{9\tau_0^{(1-k)}}{2(4\alpha + 1 - k)}\left[(\tau/\tau_0)^{1-k} H^{-4} - 1\right], \tag{3.10}$$

i.e., $T \to \infty$ for $t \to \infty$. Equation (3.5) is then

$$U_t + UU_\eta + U_{\eta\eta\eta} + \frac{A}{t}\left(\eta U_\eta + \frac{1}{2} U\right) = 0, \tag{3.11}$$

where

$$A = \frac{(2k - 9\alpha)}{6(4\alpha + 1 - k)}; \quad t = T + \frac{9\tau_0^{(1-k)}}{2(4\alpha + 1 - k)},$$

and, consequently, for large T$t \cong$ T. The energy and mass conservation laws have the forms (3.6), (3.2), respectively, with T replaced by t.

Taking into account the properties of the solution of the KdV equation discussed above, for $\sigma < \sigma_0$ one may seek a solution of Eq. (3.11) for large t in the form of series in inverse powers of t

$$U(\eta, t) = \sum_{m=0}^{\infty} U_m(\eta, t) t^{-m}. \tag{3.12}$$

For U_0 we then have the KdV equation, and for the successive approximations we have a recurrent system of linear equations. In this case U_0 has the shape of a spreading wave packet, and the absence of solitons escaping in the region of large η guarantees the smallness of terms rejected in the zeroth approximation for large t. For $\sigma > \sigma_0$, i.e., in the case of soliton formation with amplitudes a_i for the absence of terms rejected in the zeroth approximation and not decreasing with increasing t, one must represent solitons in the form

$$U_i(\eta, t) = a_i \operatorname{ch}^{-2}\left(\sqrt{\frac{a_i}{12}} Z\right), \tag{3.13}$$

where $Z = \eta - \varphi_i(t)$. The functions $\varphi_i(t)$ are found from the condition that in the η region corresponding to motion of the i-th soliton Eq. (3.11) is of the form

$$-\frac{a_i}{3} U_Z + UU_Z + U_{ZZZ} + \frac{A}{t}\left(ZU_Z + \frac{1}{2} U\right) = 0, \tag{3.14}$$

i.e., the condition that the main part of Eq. (3.14) be an equation describing a stationary soliton (in the coordinates Z, t):

$$\varphi_i = \begin{cases} \frac{a^i}{3(1-A)} t + \left(\eta_* - \frac{a_i}{3(1-A)}\right)\left(\frac{t}{t_*}\right)^A, & A \neq 1, \\ \frac{\eta_*}{t_*} t + \frac{a_i}{3} \ln \frac{t}{t_*}, & A = 1, \end{cases} \tag{3.15}$$

where t_*, η_* correspond to the moment of time and the position of the i-th soliton, only separated from the tail. Thus, in this case the amplitude, spreading, and velocity of solitons are determined by Eqs. (3.3), (3.4), and (3.15). Correction terms to the zeroth approximation, as can be seen from (3.9)-(3.12), are proportional to $1/t^m \sim [\tau^{k-1} H^4]^m$ (m = 1, 2, 3, ...), i.e., they decrease quickly with decreasing depth. We note that the KdV approximation adopted here is valid for a region of depth variation roughly 10 times the wave motion from the region of an open ocean in the direction of the continental shelf.

For k = 0 (the planar case) similar results can be obtained for the case of depth variation according to the law

$$H(\tau) = [1 - (1/L)(\tau - \tau_0)]^\alpha, \ \tau_0 \leqslant \tau \leqslant \tau_0 + L, \ \alpha > 1/4.$$

In this case all of Eqs. (3.11)-(3.15) are valid, where

$$A = \frac{9\alpha}{2(4\alpha - 1)}, \ T = \frac{9L}{2(4\alpha - 1)}\left[H^{-4+\frac{1}{\alpha}} - 1\right], \quad t = T + \frac{9L}{2(4\alpha - 1)}.$$

The additional terms are proportional to $\frac{1}{t^m} \sim H^{\frac{m(4\alpha-1)}{\alpha}}$, i.e., they decrease equally quickly (for example, for a linearly varying depth $\alpha = 1$ and $1/t^m \sim H^{3m}$). The requirement $\alpha > 1/4$ is necessary so that $T \to \infty$ for $\tau \to \tau_0 + L$. A wider class of functions $H(\tau)$, for which the procedure outlined above for obtaining asymptotics is applicable, is obviously the class of functions $H(\tau)$ allowing one to represent the expression $\tau^k H^4[(9/2)(k/\tau) + (\dot{H}/H)]$ in the form of a series in inverse powers of T. In this case $U(\eta, T)$ can be sought in a form analog to (3.12). The difference with the description above is only in the velocity of decreasing additional terms.

We note that the law of increasing wave amplitude being inversely proportional to the depth, following from (3.3) for k = 0 (planar case), was obtained in [4] by the method of adiabatic invariants.

4. Consider the case $H(\tau) \equiv 1$, k = 1. The original equation is

$$h_{0\tau} + \frac{3}{2} h_0 h_{0x} + \frac{1}{6} h_{0xxx} + \frac{1}{2\tau} h_0 = 0. \tag{4.1}$$

The analogy between Eqs. (4.1) and (3.11) is obvious. However, the problem of the asymptotic solution of Eq. (4.1) differs substantially from that considered above, since the conservation law for $h_0(\tau, x)$ is

$$\tau \int_{-\infty}^{\infty} h_0^2(\tau, x)\, dx = \tau_0 \int_{-\infty}^{\infty} h_0^2(\tau_0, x)\, dx = C, \tag{4.2}$$

and, consequently, the equation obtained by rejecting the term $h_0/2\tau$ does not possess the required conservation law (4.2), while by means of the replacement (3.3), (3.4) one achieves this correspondence between Eq. (3.11) and the conservation law (3.2). The requirement of this correspondence leads to the replacement

$$h_0(\tau, x) = \tau^{-1/2} \sum_{n=0}^{\infty} V_n t^{-n},$$

while a linear KdV equation is obtained for V_0. The asymptotic for this equation is well known [2], and is therefore not given here. Thus, the diverging axially symmetric waves for $H(\tau) \equiv 1$ are for large τ spreading wave packets with further damping $\sim\tau^{-1/2}$ in comparison with plane waves.

One can also demonstrate initial conditions for which the existence of axially symmetric waves of the soliton type is possible for large τ.

Let

$$h_0(\tau_0, x) = \mu^{-2/3} f(\mu^{-1/3} x), \quad \int_{-\infty}^{\infty} f dx = 0, \tag{4.3}$$

where $\epsilon \ll \mu \ll 1$. Substituting (4.3) into (4.2), we obtain

$$\mu\tau \int_{-\infty}^{\infty} h_0^2(\tau, x)\, dx = \tau_0 \int_{-\infty}^{\infty} f^2(z)\, dz. \tag{4.4}$$

We replace functions and independent variables

$$h_0(\tau, x) = \frac{1}{(\mu\tau)^{2/3}} U(\eta, t); \tag{4.5}$$

$$\eta = \frac{3x}{(\mu\tau)^{1/3}}, \quad t = \frac{9}{2\mu} \ln \frac{\tau}{\tau_0}. \tag{4.6}$$

From (4.4) we obtain

$$\int_{-\infty}^{\infty} U^2(\eta, t)\, d\eta = 3\tau_0 \int_{-\infty}^{\infty} f^2(z)\, dz = C. \tag{4.7}$$

Substituting (4.5), (4.6) into (4.1), we obtain

$$\frac{1}{(\mu\tau)^{5/3}} \left[U_t + UU_\eta + U_{\eta\eta\eta} - \frac{\mu}{27}\left(U + \frac{1}{2}\eta U_\eta\right)\right] = 0. \tag{4.8}$$

Representing $U(\eta, t)$ in the form of a power series of the small parameter $\mu^* = \mu/27$, we obtain for U_0 a KdV equation possessing the conservation law (4.7) required by us. According to the discussion above for $\sigma > \sigma_0$ solitons of the shape (3.13) will appear, with

$$\varphi_i = \frac{a_i}{3} + \left(\eta_* - \frac{a_i}{3}\right) e^{-\frac{1}{2}\mu^*(t - t_*)} \tag{4.9}$$

where a_i are the amplitudes of the solitons formed, and t_* and η_* correspond to the breakup moment of the soliton from the tail. It follows from (4.8), (3.13), and (4.9) that the rejected terms are of the order of μ^* for any $t \to \infty$.

We note that a similar treatment of spherical waves (k = 2), occurring, for example, in an inviscid liquid with gas bubbles, leads to the equation

$$U_t + UU_\eta + U_{\eta\eta\eta} - \mu^*\tau\left(U + \frac{1}{2}\eta U_\eta\right) = 0, \tag{4.10}$$

from which it follows that wave-type solitons can exist only for $\tau < 1/\mu^*$. From the mathematical point of view this is a consequence of the fact that unlike the axially symmetric case, for which the self-similar solution (4.5), (4.6) satisfies the energy conservation law (4.7) for the planar KdV equation, this correspondence does not exist for the spherical cases. The requirement that the zeroth approximation equation be the KdV equation and that the energy conservation law for U_0 have the shape (4.7), leads in the spherical case to the replacement

$$h_0(\tau, x) = \frac{1}{(\mu\tau)^{4/3}} U\left(\frac{x}{(\mu\tau)^{2/3}},\ t(\tau)\right),$$

a consequence of which is also Eq. (4.10).

In conclusion we note that the asymptotically diverging axially symmetric wave obtained in [5, 6] is incorrect, since for it the energy is not conserved in the wave, but increases as $\tau^{1/4}$.

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